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SYMPLECTIC STRUCTURES ON GRADED MANIFOLDS

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1. Introduction

Mixed Bose-Fermi systems are currently described by using the idea of "superspace". This concept, introduced by Salam and Strathdee [1] in the context of supersymmetry, has undergone several evolutionary phases. One of the fundamental aims to be reached by the different endeavours of generalizing and making rigorous the definition of superspace is the study of its global topological properties, whose relevance in the theory of dynamical systems cannot be overlooked. The difficulties which are encountered in working out such a program are twofold. In the first place it is necessary to give an unambiguous definition of supermanifold or graded manifold. Secondly, the definition must be flexible enough to allow for the presence of an additional structure on the graded manifold, so that a mechanical theory is actually feasible. As far as the first item is concerned, an up-to-date review of the situation together with a proposal for a new and apparently more general definition of supermanifold is given by Rogers [2]. However we find it more convenient to work in the frame of graded manifolds as defined by Kostant [3], since in our opinion their geometrical features are more thoroughly investigated, especially in the light of the requirements made in the second item. Indeed the calculus of exterior forms and cohomology rings as developed in [3] allows to speak of graded symplectic structures on graded manifolds and provides therefore the natural arena for a mechanical theory.

The purpose of the present work is to show that any graded manifold, whose underlying differentiable manifold has a symplectic structure defined by an exact 2-form [4] can itself be endowed with a graded symplectic structure. This result makes therefore rigorous the construction of mechanical models starting from a differentiable manifold (configuration space) and an arbitrary number of odd generators (independent spin variables). More

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explicitly we propose an example of such a graded symplectic manifold, and write out the canonical equations for a dynamical system described by a given hamiltonian function. Translating the geometrical language into local coordinates we obviously reproduce equations of motion analogous to those obtained in purely local theories which however rest on different principles and investigation methods [5].

2. Some notions about graded manifolds

The properties of Kostant graded manifolds are fully exposed in [3], which the reader is referred to for a detailed treatment. In this section we solely report those results which we need to study the graded symplectic structures.

Let X be a paracompact differentiable manifold of dimension m, and let $\mathcal{A}$ be a sheaf over X of Z_2-graded algebras with unity satisfying the following assumptions: if $\Gamma(U, \mathcal{A}) \equiv A(U) = A(U)_0 \oplus A(U)_1$ is the graded algebra associated with the open set $U \subseteq X$, (i.e., the algebra of the sections of $\mathcal{A}$ over U), there is an epimorphism $f \to \tilde{f}$ which maps $A(U)$ over the algebra $C^\infty(U)$ of the differentiable functions on U, graded with zero degree. Moreover for any point $p \in X$ there exists a neighborhood V in which it is defined a linear isomorphism

$$A(V) \overset{\simeq}{\to} C(V) \otimes D(V),$$

where $C(V)$, ($\equiv$ function factor), is a subalgebra of $A(V)_0$ isomorphic to $C^\infty(V)$, while $D(V)$, ($\equiv$ odd factor), is a subalgebra of $A(V)$ generated by the unity $1_{A(V)}$ and by n algebraically independent elements $s_i \in A(V)_1$, $i = 1, \cdots, n$. Any such V is called an A-splitting neighborhood of dimension n, and the sheaf $\mathcal{A}$ is said to define a graded manifold $(X, \mathcal{A})$ of dimension (m, n). An A-splitting neighborhood which is a coordinate neighborhood for the manifold X is called an A-coordinate neighborhood. The elements $r_i \in A(V)_0$, $i = 1, \cdots, m$, and $s_j \in A(V)_1$, $j = 1, \cdots, n$, are called local coordinates for $(X, \mathcal{A})$ if the functions $\tilde{r}_i$ form a local chart for X, and s_j are generators for $D(V)$.

A result of the utmost importance is the existence of a partition of the unity. This can be expressed as follows: if $U \subseteq X$ is an open set and $\mathcal{U} = \{U_h\}_{h \in H}$ is an open covering of U, there exist a locally finite refinement $\mathcal{R} = \{R_k\}_{k \in K}$ of $\mathcal{U}$ and a family $\{f_k\}_{k \in K}$ of elements of $A(U)_0$ such that: (a) $\operatorname{supp} \tilde{f}_k \subseteq R_k$; (b) $\Sigma_{k \in K} f_k = 1_{A(U)}$.

To set up the differential calculus on the graded manifold $(X, \mathcal{A})$, one introduces the sheaves $\mathcal{D}$ and $\mathcal{D}_{\mathcal{C}^\infty}$ whose sections $\Gamma(U, \mathcal{D}) = \operatorname{Der} A(U)$ and $\Gamma(U, \mathcal{D}_{\mathcal{C}^\infty}) = \operatorname{Der}(A(U), C^\infty(U))$ are respectively the derivations and the

$C^\infty(U)$-valued derivations of the algebra $A(U)$. We observe that: (a) $\mathrm{Der}\,A(U)$ is a locally free $A(U)$-module and in a chart (r_i, s_j) a basis $\mathfrak{B}$ is given by $(\partial/\partial r_i, \partial/\partial s_j)$; (b) there is a homomorphism $\xi \to \tilde{\xi}$ of $\mathrm{Der}\,A(U)$ in $\mathrm{Der}(A(U), C^\infty(U))$ such that $\tilde{\xi} f = \widetilde{\xi f}$ for any $f \in A(U)$; the local basis $\mathfrak{B}$ maps thus into a local basis $\tilde{\mathfrak{B}} = (\tilde{\partial}/\partial r_i, \tilde{\partial}/\partial s_j)$; (c) the sheaf $\mathfrak{D}_{C^\infty}$ arises from a vector bundle $T(X, \mathcal{A}) = T(X, \mathcal{A})_0 \oplus T(X, \mathcal{A})_1$, where $T(X, \mathcal{A})_0$ is isomorphic to the usual tangent bundle $T(X)$, while $T(X, \mathcal{A})_1$ is an n-dimensional vector bundle whose local basis is $(\tilde{\partial}/\partial s_j)$.

As already said, a major advantage of the theory of Kostant graded manifolds is that the sheaf of differential forms of order b, $\mathcal{E}^b$ can be naturally introduced by means of graded-alternating b-linear mappings on the derivations; indeed $\Gamma(U, E^1) = \Omega^1(U, A(U)) = \mathrm{Hom}_{A(U)}(\mathrm{Der}\,A(U), A(U))$ and $\Omega^b(U, A(U)) = \bigwedge^b \Omega^1(U, A(U))$, where $\bigwedge^b$ is the b-th graded external power. The exterior algebra $\Omega(U, A(U)) = \bigoplus_{b=0,\infty} \Omega^b(U, A(U))$ can be given the structure of differential algebra, as there is a unique external differentiation d such that $d^2 = 0$ and $\langle \xi | df \rangle = \xi f$, for $f \in \Omega^0(U, A(U)) = A(U)$ and $\xi \in \mathrm{Der}\,A(U)$. In a local chart the basis $\mathfrak{B}^*$ for $\Omega^1(U, A(U))$ dual to $\mathfrak{B}$ is denoted by (dr_i, ds_j) and we have $df = \Sigma_{i=1,m}\, dr_i \partial f/\partial r_i + \Sigma_{j=1,n}\, ds_j \partial f/\partial s_j$.

Considering the vector bundle $T^*(X, \mathcal{A})$ dual to $T(X, \mathcal{A})$, we denote by $\Omega_A(X)$ the graded exterior algebra of its sections. The natural decomposition of $T(X, \mathcal{A})$ into direct sum induces an identification of the restriction $\Omega_A(X)|_{T(X)}$ with the usual exterior algebra $\Omega(X)$ on the differentiable manifold X. The relevance of $\Omega_A(X)$ lies in the fact it is defined a projection $\beta \to \tilde{\beta}$ of $\Omega(X, A(X))$ into $\Omega_A(X)$. Moreover, if we call nondegenerate a 2-form $\omega \in \Omega^2(X, A(X))$ such that the linear mapping $\xi \to \varphi$ of $\mathrm{Der}\,A(X)$ into $\Omega^1(X, A(X))$ given by $\langle \eta | \varphi \rangle = \langle \eta, \xi | \omega \rangle$ is an isomorphism for any $\eta \in \mathrm{Der}\,A(X)$, then it is possible to show that ω is nondegenerate iff both the alternating bilinear form $\tilde{\omega}|_{T(X)}$ and $\tilde{\omega}|_{T(X,\mathcal{A})_1}$ are nondegenerate. If the mapping $\kappa\colon \Omega(X, A(X)) \to \Omega(X)$ is defined by $\kappa(\beta) = \tilde{\beta}|_{T(X)}$, we have the following commutative diagram of algebra homomorphisms:

$$\begin{array}{ccc} \Omega(X, A(X)) & \xrightarrow{\ \kappa\ } & \Omega(X) \\ & \searrow & \uparrow \\ & & \Omega_A(X) \end{array}$$

Since it is easily verified that κ commutes with the exterior differentiation d, κ is a mapping of cochain complexes and it is possible to show that the mapping $\bar{\kappa}\colon \mathrm{Coh}(\Omega(X, A(X))) \to \mathrm{Coh}(\Omega(X))$ of the de Rham cohomologies induced by κ is an algebra isomorphism.

We conclude this section by recalling that a graded symplectic manifold $(X, \mathcal{A}, \omega)$ is a graded manifold $(X, \mathcal{A})$ on which a nondegenerate closed 2-form $\omega \in \Omega^2(X, A(X))_0$ is given. As shown in [3] the usual machinery of ordinary symplectic manifolds, like Darboux theorem and Poisson algebra, can be coherently developed in the present framework.

3. An existence theorem

The relevance of symplectic manifolds in mathematical physics is due to the natural way they provide to express the hamiltonian equations of motion for a dynamical system. Indeed if ω is the 2-form defining the symplectic structure and H is the hamiltonian function, the equations of motion read

$$i_{X_H}\omega = dH.$$

The main input of a mechanical theory is usually the "configuration space", namely, a m-dimensional manifold X which constitutes the domain wherein the dynamical variables are defined. The cotangent bundle T^*X is then the symplectic manifold (phase space) where the hamiltonian function is defined. For a physical system involving n additional anticommuting variables, the configuration space is chosen to be a graded manifold $(X, \mathcal{A})$ of dimension (m, n). Again a meaning can be attached to the expression "cotangent manifold" indicating a graded manifold $(T^*X, T^*\mathcal{A})$ over T^*X of dimension $(2m, 2n)$, [3]. However, since there is the physical requirement that the odd dynamical variables are ruled by a first-order system of differential equations, [5], it appears that doubling the odd dimension has no physical foundation. It is our purpose to give a different approach indicating that a physically meaningful graded symplectic manifold can be effectively constructed out of a graded manifold $(X, \mathcal{A})$. The preliminary step is achieved by showing the following theorem.

Theorem. *Let Y be a differentiable manifold supporting a symplectic structure defined by an exact 2-form $\bar{\omega} = d\lambda$ and, let $(Y, \mathcal{B})$ be a graded manifold over Y. Then there exists an exact nondegenerate 2-form of zero Z_2-degree, $\omega \in \Omega^2(Y, B(Y))_0$, such that $\kappa\omega = \bar{\omega}$. The 2-form ω defines therefore a graded symplectic structure on $(Y, \mathcal{B})$. (Observe that any closed 2-form B such that $\kappa\beta = \bar{\omega}$ must be exact due to the isomorphism $\bar{\kappa}$ between the cohomologies.)*

Proof. Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in I}$ be a covering of Y formed by B-coordinate neighborhoods and let (r_i^α, s_j^α) be the coordinates in V_α. Since Y is assumed paracompact, we can assume without loss of generality that $\mathcal{V}$ is locally finite. Denote by $\{f_\alpha\}_{\alpha \in I}$ a partition of unity subordinate to $\mathcal{V}$. Since $(\tilde{r}_i^\alpha)$ are usual coordinate functions for V_α, we can write $\lambda|_{V_\alpha} = \Sigma_{i=1,m} \lambda_i^\alpha d\tilde{r}_i^\alpha$ with

$\lambda_i^\alpha \in C^\infty(V_\alpha)$. Define $\mu^\alpha = \Sigma_{i=1,m} dr_i^\alpha \mu_i^\alpha$ where $\mu_i^\alpha \in B(V_\alpha)_0$ is any element such that $\tilde{\mu}_i^\alpha = \lambda_i^\alpha$. Of course $\kappa\mu^\alpha = \tilde{\mu}^\alpha = \lambda|_{V_\alpha}$ in the natural identification $T(Y) \equiv T(Y, \mathfrak{B})_0$. Setting $\mu = \Sigma_{\alpha\in I} \mu^\alpha f_\alpha$, we see that $\mu \in \Omega^1(Y, B(Y))_0$ and

$$\kappa\mu = \tilde{\mu} = \sum_{\alpha\in I} \tilde{\mu}^\alpha \tilde{f}_\alpha = \sum_{\alpha\in I} \lambda|_{V_\alpha} \tilde{f}_\alpha = \sum_{\alpha\in I} \lambda \tilde{f}_\alpha = \lambda.$$

Let us now define $\nu^\alpha = \Sigma_{j=1,n} ds_j^\alpha s_j^\alpha \in \Omega^1(V_\alpha, B(V_\alpha))_0$ and $\nu = \Sigma_{\alpha\in I} \nu^\alpha f_\alpha \in \Omega^1(Y, B(Y))_0$. Finally let $\omega = d(\mu + \nu)$. Obviously $\omega \in \Omega^2(Y, B(Y))_0$ and $d\omega = 0$. Also $\kappa\omega = \overline{\omega}$ since κ commutes with the exterior differentiation. We claim that ω is nondegenerate. Indeed, according to what has been said in §2, it is sufficient to show that $\tilde{\omega}|_{T(X)}$ and $\tilde{\omega}|_{T(X,\mathcal{Q})_1}$ are nondegenerate. But $\tilde{\omega}|_{T(X)} = \kappa\omega = \overline{\omega}$ is nondegenerate by definition, while a straightforward computation shows that

$$\tilde{\omega}|_{T(X,\mathcal{Q})_1} = d\nu|_{T(X,\mathcal{Q})_1} = \sum_{\alpha\in I} \sum_{j=1,n} ds_j^\alpha ds_j^\alpha \tilde{f}_\alpha,$$

which is a riemannian metric on the vector bundle $T(X, \mathcal{Q})_1$ and therefore nondegenerate. The theorem is thus proved.

Remark. The theorem just proved explicitly requires that the 2-form $\overline{\omega}$ defining the symplectic structure on the differentiable manifold Y is exact. If we only assume $\overline{\omega}$ to be closed, then it is easy enough to show the existence of a nondegenerate 2-form $\omega \in \Omega^2(Y, B(Y))_0$. Indeed we can establish an exact sequence of sheaves $0 \to \overline{\mathcal{E}}^2 \to \mathcal{E}^2 \to \mathcal{E}_B^2 \to 0$ where $\mathcal{E}_B^2$ is the sheaf obtained by the vector bundle $\bigwedge^2 T^*(Y, \mathfrak{B})$, and $\overline{\mathcal{E}}^2$ is the appropriate kernel. Now due to the existence of the partition of unity it can be shown that $\overline{\mathcal{E}}^2$ is fine. Hence the cohomology groups $H^q(Y, \overline{\mathcal{E}}^2)$ vanish for $q > 1$, and the exact cohomology sequence reads

$$0 \to \Gamma(Y, \overline{\mathcal{E}}^2) \to \Omega^2(Y, B(Y)) \to \Omega_B^2(Y) \to 0.$$

But $\Omega_B^2(Y)_0 = \Omega^2(Y) + S^2(T^*(Y, \mathfrak{B})_1)$ where S^2 denotes the second symmetric tensor product. Since by assumption $\overline{\omega} \in \Omega^2(Y)$ is symplectic and a nondegenerate $\beta \in S^2(T^*(Y, \mathfrak{B})_1)$, i.e., a nondegenerate euclidean metric, exists, the form $\overline{\omega} + \beta$ can be lifted to $\omega \in \Omega^2(Y, B(Y))$ which is nondegenerate by the result quoted in §2. However it is by no means an easy task to decide whether ω can be chosen closed.

4. A model of graded symplectic manifold

Let $(X, \mathcal{Q})$ be a graded manifold of dimension (m, n) assumed as configuration space of a physical system. In order to describe its dynamical evolution we construct a new graded manifold of dimension $(2m, n)$ over the bundle $q: T^*X \to X$ which, according to our previous result, can be endowed with a graded symplectic structure. Intuitively what has to be done is to pull back all the generators on the cotangent bundle. This is certainly the case for an A-coordinate neighborhood $V \subseteq X$, since $A(V)$ is isomorphic to an exterior algebra over the ring $C^\infty(V)$ with n algebraically independent generators. For an arbitrary open set U this procedure cannot be applied, since $A(U)$ is not necessarily a free algebra. However it is possible to construct a graded manifold $(T^*X, \mathcal{B})$ whose local structure is just the described one, namely, $B(W) \simeq C^\infty(W) \otimes D(q(W))$ where $q(W)$ is an A-coordinate neighborhood, and $D(q(W))$ is the odd factor of $A(q(W))$. To this aim use is made of the following property concerning the structure of graded manifolds: for any graded manifold $(X, \mathcal{A})$ there exists a real vector bundle $\alpha = (E, p, X)$ such that $\mathcal{A}$ is isomorphic, as a sheaf of Z_2-graded algebras, to the sheaf determined by the sections of α.

This result has been proved by Batchelor [6] in an indirect way by exhibiting the isomorphism of the first Čech cohomology groups of X with values in the sheaves $\mathcal{A}ut(\bigwedge R^n)$ and $\mathcal{G}l(n)$, where $\Phi(U, \mathcal{A}ut(\bigwedge R^n)) = \mathrm{Aut}(C^\infty(U) \oplus \bigwedge R^n)$, and $\mathcal{G}l(n)$ is the sheaf of invertible $n \times n$ matrices with entries in $C^\infty(U)$. However this property can also be directly proved by means of the partition of the unity. Indeed in any A-coordinate neighborhood V there are isomorphisms of $C^\infty(V)$ into the function factors $C(V) \subseteq A(V)$. Let $\mathcal{V} = \{V_h\}_{h \in H}$ be a covering of X formed by A-coordinate neighborhoods, $\mathcal{R} = \{R_k\}_{k \in K}$ a locally finite refinement, $\{f_k\}_{k \in K}$ a partition of the unity subordinate to $\mathcal{R}$, and $C(R_k)$ a function factor fixed for any R_k. If $\bar{g} \in C^\infty(X)$, the restriction $\bar{g}|_{R_k} \in C^\infty(R_k)$ can be lifted to a unique element $g_k \in C(R_k)$. Let $g = \sum_{k \in K} g_k f_k \in A(X)$, and let the mapping $\sigma: C^\infty(X) \to A(X)$ be defined by $\sigma(\bar{g}) = g$. Obviously $\tilde{g} = \sum_{k \in K} \tilde{g}_k \tilde{f}_k = \bar{g}$ so that σ is an injection and $C(X) = \sigma(C^\infty(X))$ is a function factor for $A(X)$. As shown by Kostant [3], for any open set $U \subseteq X$ there is a unique function factor $C(U)$ such that $\rho_{U,X}(C(X)) \subseteq C(U)$, where $\rho_{U,X}: A(X) \to A(U)$ is the restriction mapping of the sheaf $\mathcal{A}$. (Notice that for $U = R_k$ the function factor thus defined may not coincide with that previously fixed. The correspondence $U \to C(U)$ gives a subsheaf of $\mathcal{A}$ isomorphic to C^∞). Therefore $\mathcal{A}$ can be regarded as a locally free sheaf of C^∞-modules, and hence there exists a vector bundle $\alpha = (E, p, X)$ such that $\mathcal{A}$ is the sheaf of the sections of α(7). Notice that the association $(X, \mathcal{A}) \to \alpha$ is not canonical since it depends upon the choice of $C(X)$, and it is defined only up to an isomorphism of vector bundles.

For any vector bundle $\alpha = (E, p, X)$ defined as above, we can consider the pullback $q^*\alpha = (q^*E, \bar{p}, T^*X)$ and the sheaf $\mathcal{B}$ over T^*X determined by the sections of $q^*\alpha$. $\mathfrak{B}$ can be considered a sheaf of Z_2-graded algebras and therefore produces a graded manifold $(T^*X, \mathfrak{B})$ of dimension $(2m, n)$. Again we may notice that $(T^*X, \mathfrak{B})$ is defined up to an isomorphism of graded manifolds. According to the result of §3, we denote by ω a symplectic 2-form on $(T^*X, \mathfrak{B})$. In local coordinates we can write

$$\omega = \sum_{i=1}^{m} dp_i dr_i + \frac{1}{2} \sum_{j=1}^{n} (ds_j)^2.$$

The Poisson brackets of two dynamical variables $f \in A(X)_d$ and $g \in A(X)$, $\{f, g\} = -\langle \xi_f, \xi_g | \omega \rangle$, with $\langle \xi_f, \eta | \omega \rangle = \eta f$ and $\langle \xi_g, \eta | \omega \rangle = \eta g$ for any $\eta \in \operatorname{Der} A(X)$, read

$$\{f, g\} = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial r_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial r_i} \right) + \sum_{j=1}^{n} (-1)^d \frac{\partial f}{\partial s_j} \frac{\partial g}{\partial s_j},$$

from which we deduce the usual Hamilton equations for the even variables, and $\dot{s}_j = \partial H / \partial s_j$ for the odd ones. These equations should be compared with those obtained in [5] by the use of Dirac brackets.

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